Section 3.1  Radian Measure

I.  Radian Measure
   A.  Terminology
       When a central angle (θ) intercepts the circumference of a circle, the length of the piece subtended (cut off) is called the arc length (s).
   B.  The radian measure of an angle (θ) is the ratio of the arc length (s) to the radius of the circle (r), i.e. \( \theta = \frac{s}{r} \).
   C.  If the arc length subtended by angle θ is equal to the radius (i.e., when \( s = r \)), then θ has a measure of 1 radian.
   D.  Since a radian is defined as a ratio of two lengths, the units cancel and the measure is considered unit-less. Therefore, if an angle measure is written with no degree symbol, it is assumed to be in radians. Though it is not essential, it is often customary to write radians or rads after an input measure in radians, especially when doing conversions and canceling units.
   E.  In many applications of trigonometry, radian measure is preferred over degree measure because it simplifies calculation and allows us to use the set of real numbers as the domain of the trig functions rather than just angles.

II. Converting Standard Angles Between Degrees and Radians
   A.  One full rotation measures 2π or approximately 6.28 radians. Thus 2π radians = 360°; π radians = 180°; \( \frac{\pi}{2} \) radians = 90°; \( \frac{\pi}{3} \) radians = 60°; \( \frac{\pi}{4} \) radians = 45°; and \( \frac{\pi}{6} \) radians = 30°.
   B.  It is absolutely essential that you know the equivalent degree and radian measures for all standard angles. This is not difficult if you think of every standard angle as a multiple of one of our four special angles - 30°, 45°, 60°, and 90°. For example, if \( 30^\circ = \frac{\pi}{6} \), then
       \( 60^\circ = \frac{2\pi}{6}, \ 90^\circ = \frac{3\pi}{6}, \ \text{etc.} \)
       If \( 45^\circ = \frac{\pi}{4} \), then \( 90^\circ = \frac{2\pi}{4} \); \( 135^\circ = \frac{3\pi}{4} \); and so on.

Example 1:  Convert the following angles from degree measure to radian measure.
   a.  120° (#8)    b.  225°    c.  570°
      a.  \( 120^\circ \times \frac{\pi}{180^\circ} = \frac{2\pi}{3} \)    b.  \( 225^\circ \times \frac{\pi}{180^\circ} = \frac{5\pi}{4} \)
      c.  \( 570^\circ - 360^\circ = 210^\circ \rightarrow 210^\circ \times \frac{\pi}{180^\circ} = \frac{7\pi}{6} \)
Example 2: Convert the following inputs from radian measure to degree measure.

a. \( \frac{7\pi}{4} \text{ radians} \)  
   b. \( -\frac{4\pi}{3} \text{ radians} \)  
   c. \( \frac{5\pi}{6} \text{ radians} \)

\[
a. \quad \frac{7\pi}{4} \cdot \frac{180^\circ}{\pi} = 315^\circ \\
b. \quad -\frac{4\pi}{3} \cdot \frac{180^\circ}{\pi} = -240^\circ \\
c. \quad \frac{5\pi}{6} \cdot \frac{180^\circ}{\pi} = 150^\circ
\]

C. You should memorize the following diagram.

D. You should memorize the following table.

Degree-Radian Conversion Factors

<table>
<thead>
<tr>
<th>Multiples of 30° &amp; ( \frac{\pi}{6} )</th>
<th>Multiples of 45° &amp; ( \frac{\pi}{4} )</th>
<th>Multiples of 60° &amp; ( \frac{\pi}{3} )</th>
<th>Multiples of 90° &amp; ( \frac{\pi}{2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>30° = ( \frac{1\pi}{6} )</td>
<td>45° = ( \frac{1\pi}{4} )</td>
<td>60° = ( \frac{1\pi}{3} )</td>
<td>90° = ( \frac{1\pi}{2} )</td>
</tr>
<tr>
<td>60° = ( \frac{2\pi}{6} )</td>
<td>90° = ( \frac{2\pi}{4} )</td>
<td>120° = ( \frac{2\pi}{3} )</td>
<td>180° = ( \frac{2\pi}{2} )</td>
</tr>
<tr>
<td>90° = ( \frac{3\pi}{6} )</td>
<td>135° = ( \frac{3\pi}{4} )</td>
<td>180° = ( \frac{3\pi}{3} )</td>
<td>270° = ( \frac{3\pi}{2} )</td>
</tr>
<tr>
<td>120° = ( \frac{4\pi}{6} )</td>
<td>180° = ( \frac{4\pi}{4} )</td>
<td>240° = ( \frac{4\pi}{3} )</td>
<td>360° = ( \frac{4\pi}{2} )</td>
</tr>
<tr>
<td>Angle (°)</td>
<td>Equivalent in radians</td>
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<tr>
<td>----------</td>
<td>-----------------------</td>
<td></td>
<td></td>
</tr>
<tr>
<td>150°</td>
<td>$\frac{5\pi}{6}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>225°</td>
<td>$\frac{5\pi}{4}$</td>
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<tr>
<td>300°</td>
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<td></td>
</tr>
<tr>
<td>180°</td>
<td>$\frac{6\pi}{6}$</td>
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<tr>
<td>270°</td>
<td>$\frac{6\pi}{4}$</td>
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<tr>
<td>360°</td>
<td>$\frac{6\pi}{3}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>210°</td>
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<tr>
<td>315°</td>
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<td>240°</td>
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<td>360°</td>
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<td>270°</td>
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<td>300°</td>
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<tr>
<td>330°</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>360°</td>
<td>$\frac{12\pi}{6}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### III. Converting Nonstandard Angles Between Degrees and Radians

**A.** To convert a non-standard angle from degrees to radians, multiply by $\frac{\pi}{180^o}$ and simplify.

**Example 3** Convert each nonstandard angle from degree measure to radians.

a. $174^\circ 50'$ (#40)  
b. $47.6925^\circ$ (#45)

a. $174^\circ 50' = (174 + \frac{50}{60})^\circ \approx 174.8333333\left(\frac{\pi}{180} \text{ radian}\right) \approx 3.05 \text{ radians}$

b. $47.6925^\circ = 47.6925\left(\frac{\pi}{180} \text{ radian}\right) \approx .832391 \text{ radian}$

**B.** To convert a nonstandard angle from radians to degrees, multiply by $\frac{180^o}{\pi}$ and simplify.

**Example 4** Convert each radian measure to degrees. In part b, round to the nearest minute.

a. $\frac{7\pi}{20}$ (#32)  
b. 3.06 (#50)

a. $\frac{7\pi}{20} = \frac{7\pi\left(180^\circ\right)}{20\pi} = 63^\circ$

b. 3.06 radians = $3.06 \left(\frac{180^\circ}{\pi}\right) \approx 175.3250853^\circ$

$= 175^\circ + .3250853(60') \approx 175^\circ + 19.505118' \approx 175^\circ 20'$
Example 5  Angle $\theta$ is an integer when measured in radians. Give the radian measure of the angle. (#2)

Since $\theta$ is in quadrant II, $\frac{\pi}{2} < \theta < \pi$. Since $\frac{\pi}{2} = 1.57$ and $\pi \approx 3.14$, 2 and 3 are the only integers in the interval.

Since $\theta$ is closer to $\frac{\pi}{2}$, the radian measure of $\theta$ is 2 radians.

Example 6  Find the exact value of each expression without using a calculator.

a. $\cot \frac{2\pi}{3}$ (#66)

b. $\sin \left(-\frac{5\pi}{6}\right)$

c. $\sec \frac{15\pi}{4}$

a. $\frac{2\pi}{3}$ terminates in Quadrant II and has a reference angle of $\frac{\pi}{3}$.

$$\cot \frac{2\pi}{3} = \frac{A}{O} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

b. $-\frac{5\pi}{6}$ is coterminal with $\frac{7\pi}{6}$ which terminates in Quadrant III.

It has a reference angle of $\frac{\pi}{6}$.

$$\sin \left(-\frac{5\pi}{6}\right) = \sin \frac{7\pi}{6} = \frac{O}{H} = -\frac{1}{2}$$

C. $\frac{15\pi}{4}$ is coterminal with $\frac{7\pi}{4}$ which terminates in quadrant IV.

It has a reference angle of $\frac{\pi}{4}$.

$$\sec \frac{15\pi}{4} = \sec \frac{7\pi}{4} = \frac{H}{A} = \sqrt{2}$$

Section 3.2  Applications of Radian Measure

I. Arc Length

The length (s) of the arc intercepted on a circle of radius (r) by a central angle $\theta$ (measured in radians) is given by the product of the radius and the angle, i.e., $s = r \theta$.

Caution: $\theta$ must be in radians to use this formula.

Example 1  Find the length of the arc intercepted by a central angle $\theta = 135^\circ$ in a circle of radius $r = 71.9$ cm. (#10)
Example 2  Find the distance in kilometers between Farmersville, California, 36° N, and Penticton, British Columbia, 49° N, assuming they lie on the same north-south line. The radius of the earth is 6400 km. (#14)

\[
\theta = 49^\circ - 36^\circ = 13^\circ = 13 \left( \frac{\pi}{180} \text{ radian} \right) = \frac{13\pi}{180} \text{ radian} \quad \text{and} \quad s = r\theta = 6400 \left( \frac{13\pi}{180} \right) = 1500 \text{ km}
\]

Example 3  A small gear and a large gear are meshed. An 80.0° rotation of the smaller gear causes the larger gear to rotate 50.0°. Find the radius of the larger gear if the smaller gear has a radius of 11.7 cm. (#22)

The arc length \( s \) represents the distance traveled by a point on the rim of a wheel. Since the two wheels rotate together, \( s \) will be the same for both wheels.

For the smaller wheel, \( \theta = 80^\circ = 80 \left( \frac{\pi}{180} \text{ radian} \right) = \frac{4\pi}{9} \text{ radian} \), and \( s = r\theta = 11.7 \left( \frac{4\pi}{9} \right) = 16.3363 \text{ cm} \).

For the larger wheel, \( \theta = 50^\circ = 50 \left( \frac{\pi}{180} \text{ radian} \right) = \frac{5\pi}{18} \text{ radian} \). Thus, we can solve the following.

\[
16.3363 = \frac{r \cdot 5\pi}{18} \implies r = 16.3363 \cdot \frac{18}{5\pi} = 18.720
\]

The radius of the larger wheel is 18.7 cm. (rounded to 3 significant digits)

II. Area of a Sector of a Circle

A. A sector of a circle is the portion of the interior of the circle intercepted by a central angle.

B. The area (A) of a sector of a circle of radius \( r \) with central angle \( \theta \) (measured in radians) is given by \( A = \frac{1}{2} r^2 \theta \).

Caution: \( \theta \) must be in radians to use this formula.
Example 4  Find the area of a sector of a circle with radius 18.3 cm and central angle $\theta = 125^\circ$. Round to the nearest tenth. (#36)

The formula $A = \frac{1}{2} r^2 \theta$ requires that $\theta$ be measured in radians. Converting $125^\circ$ to radians, we have

$$\theta = 125 \left( \frac{\pi}{180} \text{ radian} \right) = \frac{25\pi}{36} \text{ radians}.$$ Since $A = \frac{1}{2} (18.3)^2 \left( \frac{25\pi}{36} \right) = \frac{1}{2} (334.89) \left( \frac{25\pi}{36} \right) = 365.3083$, the area of the sector is 365.3 cm$^2$. (365.5 m$^2$ rounded to three significant digits)

Example 5  The Ford Model A, built from 1928 to 1931, had a single windshield wiper on the driver’s side. The total arm and blade was 10 inches long and rotated back and forth through an angle of $95^\circ$. If the wiper blade was 7 inches, how many square inches of the windshield did the blade clean? (#42)

The area cleaned is the area of the sector “wiped” by the total area and blade minus the area “wiped” by the arm only. We must first convert $95^\circ$ to radians.

$$95^\circ = \left( \frac{95}{180} \right) \pi \text{ radian} = \frac{19\pi}{36} \text{ radians}.$$ Since $10 - 7 = 3$, the arm was 3 in. long. Thus, we have the following.

$$A_{\text{arm only}} = \frac{1}{2} (3)^2 \left( \frac{19\pi}{36} \right) = \frac{1}{2} (9) \left( \frac{19\pi}{36} \right) = \frac{19\pi}{8} = 7.4613 \text{ in}^2.$$ and

$$A_{\text{total}} = \frac{1}{2} (10)^2 \left( \frac{19\pi}{36} \right) = \frac{1}{2} (100) \left( \frac{19\pi}{36} \right) = \frac{475\pi}{18} = 82.9031 \text{ in}^2.$$ Since $82.9031 - 7.4613 = 75.4418$, the area of the region cleaned was about 75.4 in$^2$.

Section 3.3  The Unit Circle and Circular Functions

I. Introduction

In the 1600s, scientists began using trigonometry to solve problems in physics and engineering. Such applications necessitated extending the domains of the trigonometric functions to include all real numbers, not just a set of angles. This extension was accomplished by using a correspondence between an angle and the length of an arc on a unit circle (a circle with a radius of 1, centered on the origin, with equation $x^2 + y^2 = 1$).

II. The Unit Circle

Imagine that the real number line is wrapped around a unit circle. Zero is at the point $(1, 0)$; the positive numbers wrap in a counterclockwise direction; and the negative numbers wrap in a clockwise direction. Each real number $r$ corresponds to a point $(x, y)$ on the circle. If central angle $\theta$, in standard position, measured in radians, subtends an arc length of $t$, then according to the arc length formula ($s = r \theta$), $t = 0$. 

![Unit Circle Diagram]
Thus, on a unit circle, the measure of a central angle and the length of its arc can both be represented by the same real number, \( t \).

III. The Unit Circle Definitions of the Trigonometric Functions

If \( t \) is a real number and \((x, y)\) is the point on the unit circle corresponding to \( t \), then

\[
\begin{align*}
\cos t &= x \\
\sin t &= y \\
\tan t &= \frac{y}{x}, \quad x \neq 0 \\
\sec t &= \frac{1}{x}, \quad x \neq 0 \\
\csc t &= \frac{1}{y}, \quad y \neq 0 \\
\cot t &= \frac{x}{y}, \quad y \neq 0
\end{align*}
\]

Example 1: Use the diagram below to evaluate the six circular values of \( \theta \).

![Diagram of a unit circle with a point \((\frac{\sqrt{3}}{2}, \frac{1}{2})\) and \(\theta\).]

\[
\begin{align*}
\cos \theta &= \frac{15}{17} \\
\sin \theta &= \frac{8}{17} \\
\tan \theta &= -\frac{8}{15} \\
\sec \theta &= -\frac{17}{15} \\
\csc \theta &= \frac{17}{8} \\
\cot \theta &= -\frac{15}{8}
\end{align*}
\]

IV. Domains of the Circular Functions

Since \( x = 0 \) when \( t = \frac{\pi}{2} \) and \( t = \frac{3\pi}{2} \), \( \sec t \) and \( \tan t \), which both have \( x \) in the denominator, are not defined there or at any odd multiple of \( \frac{\pi}{2} \). Thus the domain of \( \sec t \) and \( \tan t \) is \( \{ t \mid t \neq (2n + 1) \cdot \frac{\pi}{2} \} \).

Since \( y = 0 \) at \( t = 0 \), \( t = \pi \), and \( t = 2\pi \), \( \csc t \) and \( \cot t \), which both have \( y \) in the denominator, are undefined at any integer multiple of \( \pi \). Thus the domain of \( \cot t \) and \( \csc t \) is \( \{ t \mid t \neq n \cdot \pi \} \).

Sine and cosine do not have any restrictions on their domain, thus their domain is \( (-\infty, \infty) \).

V. Finding Exact Values of Circular Functions by Using Special Points on the Unit Circle

<table>
<thead>
<tr>
<th>Angle</th>
<th>0 or 2( \pi )</th>
<th>( \frac{\pi}{6} )</th>
<th>( \frac{\pi}{4} )</th>
<th>( \frac{\pi}{3} )</th>
<th>( \frac{\pi}{2} )</th>
<th>( \pi )</th>
<th>( \frac{3\pi}{2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Point</td>
<td>(1, 0)</td>
<td>( \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) )</td>
<td>( \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) )</td>
<td>( \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) )</td>
<td>(0, 1)</td>
<td>(-1, 0)</td>
<td>(0, -1)</td>
</tr>
</tbody>
</table>

The \( x \)-value of a point on the unit circle tells you \( \cos t \) and the \( y \)-value tells you \( \sin t \). The sign of the function values depends on where the angle terminates:

- **Quadrant I:** \( x \) and \( y \) are both positive, therefore **all** function values are positive.
- **Quadrant II:** \( x \) is negative and \( y \) is positive, therefore only **sine and cosecant** are positive.
- **Quadrant III:** \( x \) and \( y \) are both negative, therefore only **tangent and cotangent** are positive.
- **Quadrant IV:** \( x \) is positive and \( y \) is negative, therefore only **cosine and secant** are positive.
Example 2:  Find the exact value of (a) \( \cos \theta \), (b) \( \sin \theta \), and (c) \( \tan \theta \) for \( \theta = \frac{\pi}{2} \). (#1)

The point associated with \( \frac{\pi}{2} \) is (0, 1). Therefore, \( \cos \theta = 0; \ \sin \theta = 1; \ \tan \theta \) is undefined; \( \cot \theta = 0; \ \csc \theta = 1; \ \sec \theta \) is undefined.

Example 3:  Find the exact value of (a) \( \csc \frac{11\pi}{6} \) (#11); (b) \( \cos \frac{3\pi}{4} \) (#22); (c) \( \cot \frac{10\pi}{3} \).

a. \( \frac{11\pi}{6} \) terminates in quadrant IV. Its reference angle is \( \frac{\pi}{6} \) which is associated with the point \( \left( \frac{\sqrt{3}}{2}, -\frac{1}{2} \right) \). In quadrant IV, only cosine and secant are positive.

\[
\cos \frac{11\pi}{6} = \frac{\sqrt{3}}{2}; \quad \sin \frac{11\pi}{6} = -\frac{1}{2}; \quad \tan \frac{11\pi}{6} = -\frac{\sqrt{3}}{3}; \\
\sec \frac{11\pi}{6} = 2\sqrt{3}; \quad \csc \frac{11\pi}{6} = -2; \quad \cot \frac{11\pi}{6} = -\sqrt{3}
\]

b. \( \frac{3\pi}{4} \) terminates in quadrant II. It has a reference angle of \( \frac{\pi}{4} \) which is associated with the point \( \left( \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right) \). In quadrant II, only sine and cosecant are positive.

\[
\cos \frac{3\pi}{4} = -\frac{\sqrt{2}}{2}; \quad \sin \frac{3\pi}{4} = \frac{\sqrt{2}}{2}; \quad \tan \frac{3\pi}{4} = -1 \\
\sec \frac{3\pi}{4} = -\sqrt{2}; \quad \csc \frac{3\pi}{4} = \sqrt{2}; \quad \cot \frac{3\pi}{4} = -1
\]
c. \( \frac{10\pi}{3} \) is larger than \( 2\pi \), so we must first find its smallest positive coterminal angle.

\[
\frac{10\pi}{3} - 2\pi = \frac{4\pi}{3}
\]

which terminates in quadrant III and has a reference angle of \( \frac{\pi}{3} \).

The point associated with \( \frac{\pi}{3} \) is \( \left( \frac{1}{2}, -\frac{\sqrt{3}}{2} \right) \) and in quadrant III only tangent and cotangent are positive.

\[
\begin{align*}
\cos \frac{4\pi}{3} &= -\frac{1}{2}; \\
\sin \frac{4\pi}{3} &= -\frac{\sqrt{3}}{2}; \\
\tan \frac{4\pi}{3} &= -\frac{\sqrt{3}}{2}; \\
\sec \frac{4\pi}{3} &= -2; \\
\csc \frac{4\pi}{3} &= -\frac{2\sqrt{3}}{3}; \\
\cot \frac{4\pi}{3} &= -\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}
\end{align*}
\]

VI. Finding Approximate Values of Circular Functions

To find the approximate value of a circular function, we use a calculator set on Radian mode.

For secant, cosecant, and cotangent, we use

\[
\text{sec} \theta = \frac{1}{\cos \theta}, \quad \text{csc} \theta = \frac{1}{\sin \theta}, \quad \text{and} \quad \cot \theta = \frac{1}{\tan \theta}.
\]

Example 4: Find the approximate value of \( \sec (-8.3429) \). (#32)

\[
\sec (-8.3429) = \frac{1}{\cos (-8.3429)} \approx -2.1291
\]

VII. Determining a Number with a Given Circular Function Value

A. To determine, without a calculator, what input generates a specified output, we must be able to recognize our special point values in all their alternate forms, for example, \( \frac{2}{\sqrt{2}} = \frac{\sqrt{2}}{2} \) and \( \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \). And we must know which functions are positive in each quadrant (A S T C).

Example 5: Find the exact value of \( t \) if \( \sin t = -\frac{1}{2} \) in the interval \( \left[ \frac{\pi}{2}, \frac{3\pi}{2} \right] \).

Do not use a calculator (#58).

A sine value of \( \frac{1}{2} \) indicates the reference angle, \( r' \), is \( \frac{\pi}{3} \) and since the given interval is quadrant III, \( t \) must be \( \pi + \frac{\pi}{3} = \frac{4\pi}{3} \).

B. To determine, with a calculator, what input generates a specified output, we use an inverse trig function and the fact that if \( \text{fxn}(t) = y \); then \( \text{fxn}^{-1}(y) = t \).
Example 6  Find the values of \( t \) in \( \left[ 0, \frac{\pi}{2} \right] \) such that (a) \( \cos t = .7826 \) (\#50) and (b) \( \csc t = 1.0219 \) (\#54).

a. Since the given interval is in radians, we set the mode to RADIANS and type \( \cos^{-1}(.7826) \approx .6720 \).

b. Again, the mode should be radians, but to find \( t = \csc^{-1}(1.0219) \) we must first say \( \csc t = 1.0219 \) implies \( \frac{1}{\sin t} = 1.0219 \) and \( t = \sin^{-1}\left(\frac{1}{1.0219}\right) \approx 1.3634 \).

VIII. Applying Circular Functions

Example 7  The temperature in Fairbanks, Alaska is modeled by 
\[
T(x) = 37\sin\left(\frac{2\pi}{365}(x-101)\right) + 25
\]
where \( T(x) \) is the temperature in degrees Fahrenheit on day \( x \), with \( x = 1 \) corresponding to January 1 and \( x = 365 \) corresponding to December 31. Use the model to estimate the temperature on March 1. (\#74)

Since March 1 corresponds to day 60, 
\[
T(60) = 37\sin\left(\frac{2\pi}{365}(60-101)\right) + 25 \approx 1^\circ F.
\]

************************************************************************************

Section 3.4  Linear and Angular Speed

I. Arc Length  \( s = r \cdot \theta \)  Alternate form: \( s = r \cdot \omega \cdot t \)

For a circle with radius \( r \) and central angle \( \theta \) measured in radians, if \( \theta \) intercepts an arc length of \( s \), then \( s = r \cdot \theta \).

Hint:  For two pulleys connected by a belt or for two intermeshed gears, \( s_1 \) must equal \( s_2 \), so \( r_1 \cdot \theta_1 = r_2 \cdot \theta_2 \).

II. Linear Speed  \( v = \frac{s}{t} \)  Alternate forms: \( v = \frac{r \cdot \theta}{t} \) and \( v = r \cdot \omega \)

Consider a fly sitting on the tire of a bicycle. If we measure the linear distance (\( s \)) the fly travels per unit of time (\( t \)), we are describing its linear speed (\( v \)). Linear speed is measures in units such as miles per hour and feet per second.

III. Angular Speed  \( \omega = \frac{\theta}{t} \)  Alternate form: \( \omega = \frac{s}{r \cdot t} \)

If we measure the angle (\( \theta \)) the fly travels through per unit of time (\( t \)), we are describing its angular speed (\( \omega \)). Angular speed is measured in units such as radians per second and degrees per minute. To convert revolutions per minute to an angular speed, we multiply by \( 2\pi \) since there are \( 2\pi \) radians in one revolution. Note: The symbol for angular speed, \( \omega \), is the Greek letter omega.
IV. The Relationship between Linear Speed and Angular Speed

The linear speed \( v \) and the angular speed \( \omega \) of a point moving in a circular path with radius \( r \) are related by the formula \( v = r \cdot \omega \).

Hint: For two pulleys connected by a belt or for two intermeshed gears, \( v_1 \) must equal \( v_2 \), so \( r_1 \cdot \omega_1 = r_2 \cdot \omega_2 \).

**Example 1**
Suppose that point P is on a circle with radius \( r \), and ray OP is rotating with angular speed \( \omega \). If \( r = 30 \) cm, \( \omega = \frac{\pi}{10} \) radians per sec, and \( t = 4 \) sec, find the following. (#4)

a. the angle generated by P in time \( t \)
   \[
   \omega = \frac{\theta}{t} \Rightarrow \theta = \omega \cdot t \Rightarrow \theta = \frac{\pi}{10} \cdot 4 = \frac{2\pi}{5} \text{ radians}
   \]

b. the distance traveled by P along the circle in time \( t \)
   \[
   s = r \cdot \omega \cdot t \Rightarrow s = 30 \cdot \frac{\pi}{10} \cdot 4 = 12\pi \text{ cm}
   \]

c. the linear speed of P
   \[
   v = r \cdot \omega \Rightarrow v = 30 \cdot \frac{\pi}{10} = 3\pi \text{ cm per sec}
   \]

**Example 2**
Given \( \theta = \frac{3\pi}{8} \) radians and \( \omega = \frac{\pi}{24} \) radians per min, find \( t \). (#10)

\[
\omega = \frac{\theta}{t} \Rightarrow t = \frac{\theta}{\omega} = 0 \cdot \frac{1}{\omega} \Rightarrow t = \frac{3\pi}{8} \cdot 24 = 9 \text{ min}
\]

**Example 3**
Given \( r = 8 \) cm and \( \omega = \frac{9\pi}{5} \) radians per sec, find \( v \). (#14)

\[
v = r \cdot \omega \Rightarrow v = 8 \cdot \frac{9\pi}{5} = \frac{72\pi}{5} \text{ cm per sec}
\]

**Example 4**
Given \( s = \frac{12\pi}{5} \) m, \( r = \frac{3}{2} \) m, and \( \omega = \frac{2\pi}{5} \), find \( t \). (#22)

\[
s = r \cdot \omega \cdot t \Rightarrow t = \frac{s}{r \cdot \omega} \Rightarrow t = \frac{\frac{12\pi}{5}}{\frac{3}{2} \cdot \frac{2\pi}{5}} = \frac{12\pi}{3} \cdot \frac{5}{2\cdot 5} = 4 \text{ sec}
\]

**Example 5**
Find \( \omega \) for a line from the center to the edge of a CD revolving at 300 times per min. (#26)

\[
\omega = \frac{300 \text{ rev}}{1 \text{ min}} \cdot \frac{2\pi \text{ rad}}{1 \text{ rev}} = 600\pi \text{ radians per min}
\]
**Example 6**  
Find $v$ for a point on the tread of a tire of radius 18 cm, rotating 35 times per min. (#32)  

$$v = 18 \text{ cm} \cdot \frac{35 \text{ rotations}}{1 \text{ min}} \cdot \frac{2\pi \text{ radians}}{1 \text{ rotation}} = 1260\pi \text{ cm per min}$$

**Example 7**  
The earth revolves on its axis once every 24 hours. Assuming that Earth’s radius is 6400 km, find the following. (#38)

a. angular speed of Earth in radians per day and radians per hour  

$$\omega = \frac{1 \text{ rotation}}{1 \text{ day}} \cdot \frac{2\pi \text{ radians}}{1 \text{ rotation}} = 2\pi \text{ radians per day}$$

$$\omega = \frac{2\pi \text{ radians}}{1 \text{ day}} \cdot \frac{1 \text{ day}}{24 \text{ hours}} = \frac{\pi}{12} \text{ radians per hour}$$

b. linear speed at the North Pole or South Pole  
Since $r = 0$ at the North or South Pole, $v = 0$ at the North or South Pole.

c. linear speed at Quito, Ecuador, a city on the equator  
At the equator, $r = 6400$ km, so  

$$v = 6400 \cdot \frac{\pi}{12} \approx 533\pi \text{ km per hour}$$

or  

$$v = 6400 \cdot \frac{\pi}{12} \approx 533\pi \text{ km per hour}$$

d. linear speed at Salem, Oregon (halfway from the equator to the North Pole)  
Since Salem is halfway between the equator and the North Pole, if we draw a radius from the center of the earth to Salem, it forms a 45° angle with the equator. Thus  

$$\sin 45^\circ = \frac{r}{6400} \rightarrow r = 6400 \cdot \sin 45^\circ$$

$$r = 6400 \cdot \frac{\sqrt{2}}{2} = 3200\sqrt{2} \approx 4525.4834 \text{ km}$$

Therefore,  

$$v = 3200\sqrt{2} \cdot 2\pi = 6400\pi\sqrt{2} \approx 28434.4508 \text{ km per day}$$

or  

$$v = 3200\sqrt{2} \cdot \frac{\pi}{12} = \frac{800\pi\sqrt{2}}{3} \approx 1184.768784 \approx 1200 \text{ km per hr}$$